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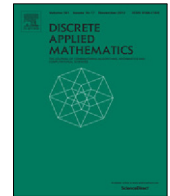
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Characterizing graphs of maximum matching width at most 2

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ABSTRACT

The maximum matching width is a width-parameter that is defined on a branch-decomposition over the vertex set of a graph. The size of a maximum matching in the bipartite graph is used as a cut-function. In this paper, we characterize the graphs of maximum matching width at most 2 using the minor obstruction set. Also, we compute the exact value of the maximum matching width of a grid.

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1. Introduction

Treewidth and branchwidth are well-known width-parameters of graphs used in structural graph theory and theoretical computer science. Based on Courcelle's theorem [4], which states that every property on graphs definable in monadic second-order logic can be decided in linear time on a class of graphs with bounded treewidth, many NP-hard problems have been shown to be solvable in polynomial time by dynamic programming when the input has bounded treewidth or branchwidth.

Vatshelle [20] introduced a new graph width-parameter, called the maximum matching width (mm-width in short), that uses the size of a maximum matching as a cut-function in its branch-decomposition of the vertex set of a graph. Maximum matching width is related to treewidth and branchwidth as shown by the inequality $\text{mmw}(G) \leq \max(\text{brw}(G), 1) \leq \text{tw}(G) + 1 \leq 3 \text{mmw}(G)$ for every graph G [20] where $\text{mmw}(G)$, $\text{tw}(G)$, and $\text{brw}(G)$ are the maximum matching width, the treewidth, and the branchwidth of G respectively. This implies that bounding the treewidth or branchwidth is qualitatively equivalent to bounding the maximum matching width. Maximum matching width gives a more efficient algorithm for some problems. For a given branch-decomposition of a graph G of maximum matching width k , we can solve the Minimum Dominating Set Problem in time $O^*(8^k)$ [8], which gives a better runtime than $O^*(3^{\text{tw}(G)})$ -time algorithm in [19] when $\text{tw}(G) > (\log_3 8)k$. Note that the Minimum Dominating Set Problem cannot be solved in time $O^*((3 - \varepsilon)^{\text{tw}(G)})$ for every $\varepsilon > 0$ unless the Strong Exponential Time Hypothesis fails [10].

The Robertson–Seymour theorem [13] states that every minor-closed class of graphs has a finite minor obstruction set. In the other words, a graph G is in the class if and only if G has no minor isomorphic to a graph in the obstruction set. Much work has been done to identify the minor obstruction set for various graph classes, especially for graphs of bounded width-parameters [2,5,9].

Let K_n , C_n , and P_n be the complete graph, the cycle graph, and the path graph on n vertices, respectively. The graphs K_3 and K_4 are the unique minor obstruction for the graphs of treewidth at most 1 and 2 [21], respectively. The minor obstruction set for the class of graphs having treewidth at most 3 is $\{K_5, K_{2,2,2}, K_2 \times C_5, M_8\}$ where $K_2 \times C_5$ is the Cartesian product of K_2 and C_5 , and M_8 is the Wagner graph, also called the Möbius ladder with eight vertices [1,16].

Robertson and Seymour [12] gave a characterization for the classes of graphs of branchwidth at most 1 and at most 2. The graphs K_3 and P_4 are forbidden minors for the graphs of branchwidth at most 1. For the class of graphs of branchwidth

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at most 2, its minor obstruction is the same as treewidth, which is K_4 . The graphs of branchwidth at most 3 have four minor obstructions; $\{K_5, K_{2,2,2}, K_2 \times C_4, M_8\}$ [3].

One of the main results of this paper is to find the minor obstruction set for the class of graphs of mm-width at most 2. Note that the class of graphs with bounded mm-width is closed under taking minor, as shown in Corollary 2.3. Our main result is the following.

Theorem 3.17. *Let $\mathcal{O} = \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$ be the set of 42 graphs in Figs. 1, 5–7. A graph G has mm-width at most 2 if and only if G has no minor isomorphic to a graph in \mathcal{O} .*

The exact value of some width-parameters for grid graphs is well known. For an integer $k \geq 1$, the branchwidth and treewidth of the $k \times k$ -grid are k [12,17], and the rank-width of the $k \times k$ -grid is $k - 1$ [7]. From the inequality $\text{rw}(G) \leq \text{mmw}(G) \leq \max(\text{brw}(G), 1)$ [20], the mm-width of the $k \times k$ -grid is either $k - 1$ or k . Our second result is that the latter is the right answer when $k \geq 2$.

Theorem 4.7. *The $k \times k$ -grid has mm-width k for $k \geq 2$.*

Section 2 lists some of the definitions, including a *tangle*, and provides preliminaries for the maximum matching width. In Section 3 we identify the minor obstruction set for graphs with mm-width at most 2. Section 4 gives the result for the precise mm-width of the square grids.

2. Preliminaries

Every graph $G = (V, E)$ in this paper is finite and simple. For a set $X \subseteq V(G) \cup E(G)$, we write $G \setminus X$ to denote the graph obtained from G by deleting all vertices and edges in X . If $X \subseteq E(G)$, we write G/X to denote the graph obtained from G by contracting the edges in X . If $X = \{x\}$, then we write $G \setminus x$ and G/x instead of $G \setminus X$ and G/X , respectively. If a subgraph G' of G with $V(G') = X$ contains all the edges of G whose both ends are in X , then we call G' *induced by X* and write $G' := G[X]$. For a graph G and disjoint subsets $X, Y \subseteq V(G)$, let $E_G[X, Y]$ (or $E[X, Y]$) denote the set of all edges $e = uv$ where u is in X and v is in Y , and let $G[X, Y] = G(X \cup Y, E[X, Y])$. A graph G is *k-connected* if $|V(G)| \geq k$ and $G \setminus X$ is connected for every $X \subset V(G)$ with $|X| < k$. A *bridge* is an edge e such that $G \setminus e$ has more components than G . A *block* is either a bridge as a subgraph or a maximal 2-connected subgraph.

We say that a tree is *ternary* if all vertices have degree 1 or 3. A *branch-decomposition* of a finite set X is a pair (T, \mathcal{L}) of a ternary tree T together with a bijection \mathcal{L} from the leaves of T to X . Note that an edge ab of T partitions the leaves of T into two parts, say A and B . We say an edge e *induces* the partition (A, B) . A function $f : 2^X \rightarrow \mathbb{Z}$ is *symmetric* if $f(A) = f(X \setminus A)$ for all $A \subseteq X$, and the function f is *submodular* if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq X$. For each edge e of T , and a symmetric, submodular function f , the *f-value* of e is equal to $f(A) = f(B)$ where (A, B) is the partition induced by e . The *f-width* of a branch-decomposition (T, \mathcal{L}) is the maximum f -value of an edge of T , and the *f-width* of X is the minimum value of the f -width over all possible branch-decompositions of X . This notion of f -width provides a link between several width parameters.

For $A \subseteq E(G)$, let $br : 2^{E(G)} \rightarrow \mathbb{Z}$ be the function so that $br(A)$ is the number of vertices that are incident to both an edge in A and an edge in $E(G) \setminus A$. The *branchwidth* of G , denoted by $\text{brw}(G)$, is the *br-width* of $E(G)$.

For $A \subseteq V(G)$, let $r : 2^{V(G)} \rightarrow \mathbb{Z}$ be the function such that $r(A)$ is the rank of the adjacency matrix between A and $V(G) \setminus A$ over \mathbb{F}_2 . The *rank-width* of G , denoted by $\text{rw}(G)$, is the *r-width* of $V(G)$.

Let $\text{mm}_G : 2^{V(G)} \rightarrow \mathbb{Z}$ be the function such that $\text{mm}_G(A)$ is the size of a maximum matching in $G[A, V(G) \setminus A]$. Note that the function mm_G is symmetric and submodular [15]. We use mm instead of mm_G if the host graph G is clear from the context. The *maximum matching width* of G , denoted by $\text{mmw}(G)$, is the mm-width of $V(G)$.

A graph H is a *minor* of a graph G if H can be constructed from G by deleting edges, deleting vertices, and contracting edges. We call a graph G *minor-minimal* with respect to a property \mathcal{P} if G has \mathcal{P} but no proper minor of G has \mathcal{P} . A graph G is a *forbidden minor* of a graph class \mathcal{C} if $H \notin \mathcal{C}$ whenever H has a minor isomorphic to G . Robertson and Seymour [13] state that the collection of minor-minimal graphs outside a minor-closed graph class is finite. The collection is called the *minor obstruction set*.

A graph is *chordal* if every induced cycle in the graph has length 3. A *chordalization* of a graph G is a chordal graph H such that $V(H) = V(G)$ and $E(G) \subseteq E(H)$. An *intersection graph* G over a family $\{A_i\}$ of sets is the graph with $V(G) = \{A_i\}$ and $E(G) = \{A_i A_j : A_i \cap A_j \neq \emptyset\}$. Remark that a graph is chordal if and only if it is the intersection graph of the edge sets of subtrees of a tree [6].

2.1. Maximum matching width

Jeong, Sæther, and Telle [8] gave a new characterization of graphs of mm-width at most k as an intersection graph by the following theorem. A tree is called *nontrivial* if it has at least one edge and a tree is *subcubic* if all vertices have degree at most 3.

Theorem 2.1 ([8]). *The maximum matching width of a graph G is at most k if and only if there exist a subcubic tree T and a set $\{T_x\}_{x \in V(G)}$ of nontrivial subtrees of T such that*

- (1) if $uv \in E(G)$, then the subtrees T_u and T_v have at least one vertex of T in common,
- (2) for each edge e of T there are at most k subtrees in $\{T_x\}_{x \in V(G)}$ containing e .

Note that we can replace a subcubic tree T to a ternary tree T in [Theorem 2.1](#) because we can add an edge to each degree-2 vertex of T .

A tree-representation of G having width at most k is a pair $(T, \{T_x\}_{x \in V(G)})$ where T is a ternary tree and a set $\{T_x\}_{x \in V(G)}$ of nontrivial subtrees satisfying the properties (1) and (2). [Theorem 2.1](#) says that a graph G has a tree-representation of width at most k if and only if $\text{mmw}(G) \leq k$.

For a tree-representation $(T, \{T_x\}_{x \in V(G)})$ of G , the intersection graph G_T of the family $\{T_x\}_{x \in V(G)}$ is chordal and G is a subgraph of G_T . Since G and G_T have the same tree-representation $(T, \{T_x\}_{x \in V(G)})$, every graph has a chordalization with the same mmw -width.

It is easy to check that, for a graph G and its vertex or edge x ,

$$\text{mmw}(G \setminus x) \leq \text{mmw}(G).$$

The same holds for contracting an edge.

Lemma 2.2. Let G be a graph. For every edge uv of G , $\text{mmw}(G/uv) \leq \text{mmw}(G)$.

Proof. Let $(T, \{T_x\}_{x \in V(G)})$ be a tree-representation of G having width $\text{mmw}(G)$. Let T_{uv} be the subtree of T with vertex set $V(T_u) \cup V(T_v)$ and edge set $E(T_u) \cup E(T_v)$. Then $(T, \{T_x\}_{x \in V(G) \setminus \{u,v\}} \cup \{T_{uv}\})$ is a tree-representation of G/uv having width at most $\text{mmw}(G)$. By [Theorem 2.1](#), $\text{mmw}(G/uv) \leq \text{mmw}(G)$. \square

Corollary 2.3. Let k be an integer. The set $M_k = \{G : \text{mmw}(G) \leq k\}$ is closed under the minor operations.

By [Corollary 2.3](#) and Robertson–Seymour theorem [13], M_k has a finite minor obstruction set for each k . We can easily find the minor obstruction set when $k = 1$.

Proposition 2.4 ([14]). A graph G has mmw -width at most 1 if and only if G does not contain C_4 as a minor.

Proof. Suppose that G contains C_4 as a minor. We can find four vertices v_1, v_2, v_3, v_4 of G and four paths $P_{12}, P_{23}, P_{34}, P_{41}$ in G such that each path P_{ij} is a path from v_i to v_j and the four paths are pairwise internally vertex-disjoint. For every branch-decomposition (T, \mathcal{L}) of $V(G)$, there exists an edge e in T that induces a partition (A, B) of $V(G)$ such that two vertices from v_1, v_2, v_3, v_4 are in A and the other two are in B . Thus, there exist two vertex-disjoint paths from A to B . This implies that the mm_G -value of e is at least 2, and therefore G has mmw -width at least 2.

Now let us suppose that G does not contain C_4 as a minor. It is easy to see that every block of G is either K_2 or C_3 . The mmw -width of G is the maximum value among the mmw -widths of blocks of G . Since both K_2 and C_3 have mmw -width 1, G has mmw -width at most 1. \square

2.2. Tangle

Before proving our main theorems, we shall introduce the notion of *tangle*, which is useful in investigating the lower bounds of width-parameters.

Let f be an integer-valued symmetric submodular function on the subsets of a finite set X . An f -tangle of order $k + 1$ is a collection \mathcal{T} of subsets of X satisfying that

- (T1) for all $S \subseteq X$, if $f(S) \leq k$, then one of S and $X \setminus S$ is in \mathcal{T} ,
- (T2) if $S_1, S_2, S_3 \in \mathcal{T}$, then $S_1 \cup S_2 \cup S_3 \neq X$,
- (T3) for each $x \in X$, $X \setminus \{x\} \notin \mathcal{T}$.

Robertson and Seymour [12] proved the following theorem. We use it in both Sections 3 and 4.

Theorem 2.5 ([12]). Let f be an integer-valued symmetric submodular function on subsets of a finite set X . The f -width of X is larger than k if and only if there exists an f -tangle of order $k + 1$.

The $k \times k$ -grid, denoted by G_k , is the graph with vertex set $V(G_k) = \{(i, j) : 1 \leq i, j \leq k\}$ and edge set $E(G_k) = \{(i, j)(i', j') : |i - i'| + |j - j'| = 1\}$. Using [Theorem 2.5](#), we show that the 3×3 -grid has mmw -width 3, as an example.

Lemma 2.6. The 3×3 -grid G_3 has an mmw -tangle of order 3.

Proof. Let us consider G_3 to be a part of an integer grid in the real plane and let $\{(i, j) : 1 \leq i, j \leq 3\}$ be the vertex set of G_3 . Let A be a set of all subsets of $V(G_3)$ with size at most 2. Let $B = \{(1, 1), (1, 2), (2, 1)\}, \{(1, 2), (1, 3), (2, 3)\}, \{(2, 3), (3, 2), (3, 3)\}, \{(2, 1), (3, 1), (3, 2)\}$. We claim that $A \cup B$ is an mmw -tangle of order 3. It is trivial that (T3) holds. If $S_1 \cup S_2 \cup S_3 = V(G_3)$,

then the sets S_1, S_2, S_3 must be in B . However, no set in B has $(2, 2)$ and thus (T2) follows. Now we check (T1). Note that for every subset $S \subseteq V$ with $|S| = 4$, we have $\text{mm}(S) \geq 3$. Since A contains all subsets of size at most 2, we need to consider subsets of $V(G_3)$ of size 3. The elements in B are the only subsets of size 3 having $\text{mm}(S) \leq 2$. Hence (T1) holds too and $A \cup B$ is a mm-tangle of order 3. \square

By Lemma 2.6 and Theorem 2.5, the 3×3 -grid has mm-width at least 3. It is easy to see that the 3×3 -grid has mm-width at most 3 since it has 9 vertices and K_9 has a tree-representation of width 3. Thus the 3×3 -grid has mm-width 3. In this paper, we use a similar argument to verify that the graphs in the minor obstruction set for mm-width at most 2 have mm-width 3. Note that the 3×3 -grid is also in the minor obstruction set for the graphs of mm-width at most 2. See Fig. 6(b).

3. Minor obstruction set for maximum matching width at most 2

Note that if G is not 2-connected, then $\text{mmw}(G)$ is the maximum of $\text{mmw}(H)$ where H is a maximal 2-connected subgraph of G . Thus the graphs in the minor obstruction set are 2-connected.

In Section 3.1 we identify the 3-connected graphs that are minor-minimal with respect to $\text{mm-width} \geq 3$. And then we consider the minor-obstructions with 2-cuts in Section 3.2. We shall show that each 2-cut separates the graph into at most three components, where all but one component is small (a full characterization is given after Lemma 3.7). We show that the obstructions are obtained from a 3-connected graph with ≤ 6 vertices by replacing some edges with small components mentioned above. What remains is to check all the candidates.

3.1. 3-connected graphs

In this subsection, we give five 3-connected graphs that have mm-width 3 and whose proper minors have mm-width 2.

Lemma 3.1. *For each ternary tree T , one can find an edge of T whose removal divides the set of leaves into two subsets, each having at least $1/3$ of all the leaves.*

Proof. Let $e = uv$ be an edge that induces a partition (A, B) of the leaves where u is on the side of A . Suppose that A contains more than $2/3$ of the leaves. Then u has degree 3 and the other two edges at u induce leaf partitions, namely (A_1, B_1) and (A_2, B_2) where we assume u to be on the side of A_1 and A_2 respectively. We choose the edge, say e' , with larger $|A_i|$. If both A_1 and A_2 contain at most $2/3$ of the leaves then e' will be the edge we are after. Otherwise, we have a partition with smaller difference $|A_i| - |B_i|$ than $|A| - |B|$ and we iterate until we find a working edge. \square

By Lemma 3.1, a ternary tree with at least 7 leaves has an edge dividing the leaves into two sets such that both have size at least 3.

Lemma 3.2. *If a graph G is 3-connected and G has at least 7 vertices, then $\text{mmw}(G) \geq 3$.*

Proof. By the argument above, for every branch decomposition (T, \mathcal{L}) of $V(G)$, we can find an edge e in T inducing a partition (A, B) with $|A|, |B| \geq 3$. Since G is 3-connected, by Menger's theorem, G has three vertex-disjoint paths between A and B . These paths give a matching of size 3 in $G[A, B]$, which means that the mm_G -value of e is at least 3. Thus, every branch-decomposition of $V(G)$ has mm_G -width at least 3. \square

It is easy to find a tree-representation of K_{3n} with width n . In particular, K_6 has mm-width 2 and hence every graph on 6 vertices has mm-width at most 2. In other words, the forbidden minors for mm-width at most 2 have at least 7 vertices. We use Tutte's wheel theorem stated below. In the following statement we assume pairwise parallel edges occurring from contractions are all removed but one to keep the graph simple.

Theorem 3.3 (Tutte's Wheel Theorem [18]). *If a graph G is 3-connected, then G has an edge e such that either G/e or $G \setminus e$ is 3-connected unless $G = K_4$.*

Lemma 3.4. *Let \mathcal{O}_3 be the set of the five graphs in Fig. 1. A 3-connected graph is minor-minimal with respect to maximum matching width at least 3 if and only if it is in \mathcal{O}_3 .*

Proof. By Tutte's wheel theorem, a 3-connected graph with at least 8 vertices has a proper 3-connected minor with at least 7 vertices, which has mm-width at least 3 by Lemma 3.2. Thus a minor-minimal 3-connected graph with respect to mm-width at least 2 has precisely 7 vertices. By [11], the five graphs in Fig. 1 are precisely the edge-minimal 3-connected graphs on 7 vertices, and hence it is enough to show that the proper minors of these graphs all have mm-width at most 2.

Observe that all edges of a graph in \mathcal{O}_3 are incident with a vertex of degree 3. Thus by taking out the edge we have a graph on 7 vertices with at least one vertex of degree 2, say v . Starting from a tree-representation of $G \setminus v$ with width 2, by rearranging the leaves if needed, we can easily add a vertex v of degree 2 without increasing mm-width, so such a graph must have mm-width 2. \square

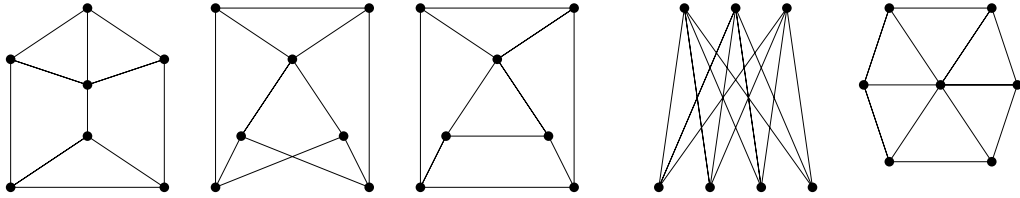


Fig. 1. The minor-minimal 3-connected graphs on 7 vertices.

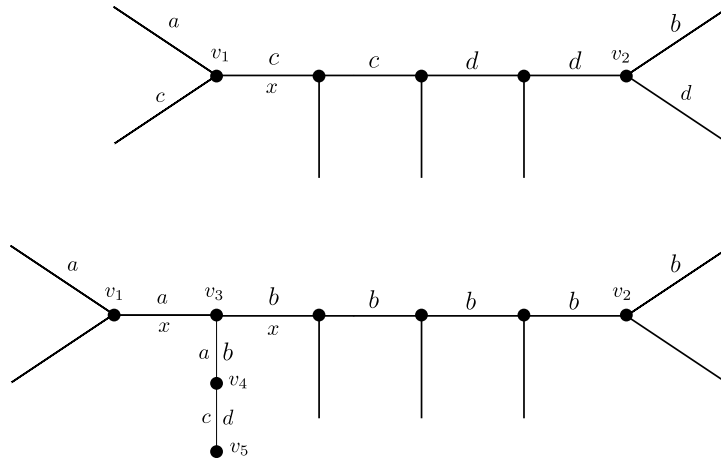


Fig. 2. Another tree-representation of the same graph where T_a and T_b share an edge.

3.2. 2-connected graphs

Now we find 2-connected minor-minimal graphs with respect to mm-width 3 that are not 3-connected. Let \mathcal{O}_2 be the set of all graphs G such that G is not 3-connected and G is minor-minimal with respect to mm-width at least 3. Note that the graphs in \mathcal{O}_2 are 2-connected.

We say that a tree-representation of G is *good* if there exist two vertices a and b such that the subtrees for a and b share an edge and the width of the tree-representation is 2. A pair $(G, \{a, b\})$ is *good* if it has a corresponding good tree-representation with vertices a and b , and *bad* if none exists.

Lemma 3.5. Let G be a graph and let $a, b \in V(G)$. Let H be the graph obtained from G by adding two new vertices, say c and d , and edges ac , cd and db , followed by removing the edge ab if $ab \in E(G)$. If $(G, \{a, b\})$ is bad, then $\text{mmw}(H) \geq 3$.

Proof. We prove by contradiction. Suppose $\text{mmw}(H) \leq 2$, that is, H has a tree-representation $\mathcal{T} = (T, \{T_v\}_{v \in V(H)})$ of width at most 2. We shall use \mathcal{T} to find a good tree-representation of G with a and b , yielding a contradiction.

From \mathcal{T} we may obtain three tree-representations of G with width at most 2 by replacing the subtree for a and b respectively with (1) $T_a \cup T_c \cup T_d$ and T_b , (2) $T_a \cup T_c$ and $T_d \cup T_b$, and (3) T_a and $T_c \cup T_d \cup T_b$. Since $(G, \{a, b\})$ is bad, for all three choices the subtrees for a and b intersect at precisely one vertex in the new tree-representations. Therefore, $E(T_a) \cap E(T_b) = \emptyset$ and T has two distinct vertices v_1 and v_2 such that $T_a \cap T_c = \{v_1\}$ and $T_d \cap T_b = \{v_2\}$.

Let $e = v_1u$ be the first edge in the unique path P in T from v_1 to v_2 . Because of the path $acdb$ in H , the first few consecutive edges of P (possibly zero) are in T_c and the others are in T_d . We assume that T_c contains e . The following manipulation can be done likewise when T_d contains e .

Let $x \neq c$ be a vertex of G such that T_x contains e . If there is no such x we ignore x in the following. Let T' be the tree obtained from T by subdividing e , and adding a path $v_3v_4v_5$ of length 2 at the new vertex v_3 obtained from the subdivision; see Fig. 2. Let $\{T'_v\}_{v \in V(H)}$ be a collection of subtrees of T' such that

- T'_c and T'_d have only one edge v_4v_5 ,
- T'_a is obtained from T_a by adding the edges v_1v_3 and v_3v_4 ,
- T'_b is obtained from T_b by adding v_3v_4 and the edges on the path from v_3 to v_2 , and
- $T'_v = T_v$ for all other $v \in V(H)$.

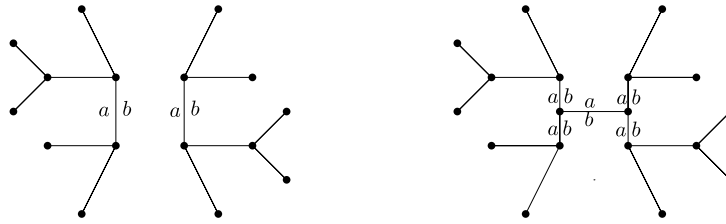


Fig. 3. New tree-representation from two tree-representations.

Note that for each pair of vertices $u, v \in V(H) \setminus \{c, d\}$, if $T_u \cap T_v \neq \emptyset$ then $T'_u \cap T'_v \neq \emptyset$. Since c and d are adjacent to only a and b in H , the pair $\mathcal{T}' = (T', \{T'_v\}_{v \in V(H)})$ is a tree-representation of H of width 2. Hence, by removing v_5 , we obtain a good tree-representation of G with a, b having width 2, a contradiction. \square

Lemma 3.6. Let G be a graph and let c be a vertex of G with precisely two neighbors a and b . If $ab \in E(G)$ and $\text{mmw}(G) \geq 3$, then $\text{mmw}(G \setminus ab) \geq 3$.

Proof. We prove by contradiction. Suppose $H = G \setminus ab$ has a tree-representation $\mathcal{T} = (T, \{T_v\}_{v \in V(H)})$ of width at most 2. Since $\text{mmw}(G) \geq 3$ the subtrees T_a and T_b are vertex-disjoint. Let $v_1 \in V(T_a)$ and $v_2 \in V(T_b)$ be the vertices of T such that the unique path P in T from v_1 to v_2 have no edge in neither T_a nor T_b . As c is a common neighbor of a and b , every edge of P is in T_c . Now we do the same as in the proof of Lemma 3.5 and Fig. 2, except that here we set $d = c$. The resulting tree-representation of G has width at most 2, a contradiction. \square

A 2-cut in G is an inclusion-wise minimal subset $S \subset V(G)$ such that $|S| = 2$ and $G \setminus S$ is disconnected. Given a graph G and its 2-cut $\{a, b\}$ with a component S of $G \setminus \{a, b\}$, we denote by \tilde{S} the induced subgraph $G[V(S) \cup \{a, b\}]$. As $\{a, b\}$ is the unique 2-cut having S as a component, we may say simply \tilde{S} is good or bad, if respectively the pair $(\tilde{S}, \{a, b\})$ is good or bad.

Lemma 3.7. Let G be a graph in \mathcal{O}_2 . If a 2-cut $\{a, b\}$ separates G into two components A and B , then $ab \notin E(G)$ and one of \tilde{A} or \tilde{B} is isomorphic to either P_3 or P_4 .

Proof. We start with showing that one of \tilde{A} and \tilde{B} is bad. Suppose for contradiction that both are good. From their good representations, say $(T^A, \{T_x\}_{x \in V(A)})$ and $(T^B, \{T_y\}_{y \in V(B)})$, we can construct a tree-representation of G of width 2 as follows. We choose an edge from each of T^A and T^B shared by T_a and T_b , and then subdivide those two edges and connect the new vertices by an edge; see Fig. 3. The new subtrees T'_a and T'_b will be clear from Fig. 3. It is easy to see that the resulting tree-representation has width 2.

Now we assume \tilde{B} is bad. Suppose $|A| \geq 2$. Since G is 2-connected, there exists a path from a to b in \tilde{A} whose length is at least 3. Thus, \tilde{A} contains P_4 as a minor. Let H be a graph obtained from \tilde{B} and P_4 by identifying a and b in \tilde{B} with two ends of P_4 respectively. By Lemma 3.5, H has mm-width at least 3. If P_4 is a proper minor of \tilde{A} , then a graph H is a proper minor of G . It contradicts that G is in \mathcal{O}_2 . Therefore, \tilde{A} is isomorphic to P_4 .

Suppose $A = \{c\}$. Since G is 2-connected, c is adjacent to both a and b and by Lemma 3.6, \tilde{A} is the path acb , which is isomorphic to P_3 . \square

To consider the 2-cuts with more than two components, we use the following lemma. The proof of Lemma 3.5 can be modified to prove the following.

Lemma 3.8. Let G be a graph and let $a, b \in V(G)$. Let H be the graph obtained from G by adding two new vertices, say c and d , and edges ac, bc, ad and bd , followed by removing the edge ab if $ab \in E(G)$. If $(G, \{a, b\})$ is bad, then $\text{mmw}(H) \geq 3$.

Let $G \in \mathcal{O}_2$. Suppose that a 2-cut $\{a, b\}$ separates G into at least three components, namely D_1, D_2, \dots, D_k . Since we can combine arbitrary number of good tree-representations as in Fig. 3 while preserving goodness, one of the \tilde{D}_i 's, say \tilde{D}_1 , is bad. If $k > 3$, we construct G' by contracting each of D_2, \dots, D_k into a single vertex, and leaving just two of those. Then by Lemma 3.8, $\text{mmw}(G') \geq 3$. Therefore, G cannot contain G' as a proper minor. Also by Lemma 3.7, \tilde{D}_1 is isomorphic to either P_3 or P_4 for $2 \leq i \leq k$. As a result, one of the following holds:

1. $k = 2$ and $\tilde{D}_2 = P_3$.
2. $k = 2$ and $\tilde{D}_2 = P_4$.
3. $k = 3$ and $D_2 = \tilde{D}_3 = P_3$.

We summarize the above discussion as follows.

Corollary 3.9. Let G be a graph in \mathcal{O}_2 . If $\{a, b\}$ is a 2-cut of G , then $G \setminus \{a, b\}$ has a unique component B_{ab} such that $\tilde{B}_{ab} = G[V(B_{ab}) \cup \{a, b\}]$ is bad. Moreover, $ab \notin E(G)$ and $G \setminus B_{ab}$ is one of the following:

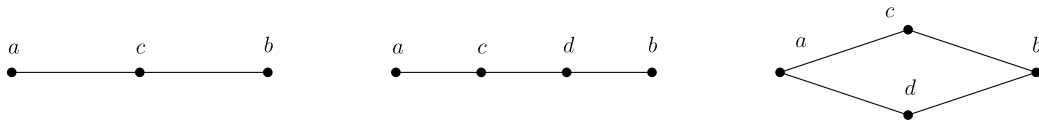


Fig. 4. Three ways of replacing an edge ab .

- a path of length 2 between a and b ,
- a path of length 3 between a and b , or
- a $K_{2,2}$ where a and b are non-adjacent.

Following Corollary 3.9, for each 2-cut $\{a, b\}$ of $G \in \mathcal{O}_2$, we call $G \setminus B_{ab}$ the *good-side* of $\{a, b\}$. Hence the good-side of $\{a, b\}$ contains at most two vertices other than a and b .

We shall show below that every graph in \mathcal{O}_2 can be constructed from a small 3-connected graph by replacing some of its edges by some of the three graphs in Fig. 4. To state this precisely, we call the replacement of an edge ab with $P_3 = acb$, $P_4 = acdb$ and $K_{2,2} = acb \cup adb$, respectively, as *1-subdivision*, *2-subdivision* and *11-subdivision* where c, d are adjacent to no other vertices; see Fig. 4. We call these three operations as *good-subdivisions*.

Lemma 3.10. Every graph in \mathcal{O}_2 is obtained from a 3-connected graph on 4, 5, or 6 vertices by good-subdividing some of its edges.

Proof. Let us consider the inclusion-wise maximal good-sides of 2-cuts. We would like to replace each of them with an edge between the vertices in its 2-cut. To make this operation valid, we begin with showing that if two good-sides intersect, then both of them are contained in a good-side that is P_4 , or the intersection is a single vertex contained in both of their 2-cuts. Note that if \tilde{A} is bad then A has at least 5 vertices, as K_6 has a good tree-representation for every pair of its vertices.

Let G be a graph in \mathcal{O}_2 . Let $\{a, b\}$ be a 2-cut in G and let B_{ab} be the unique component of $G \setminus \{a, b\}$, by Corollary 3.9, such that $\tilde{B}_{ab} = G[V(B_{ab}) \cup \{a, b\}]$ is bad. Let $\{c, d\}$ be another 2-cut in G and suppose that c is in $G \setminus \tilde{B}_{ab}$. By Corollary 3.9, c has only two neighbors in G and is in a path connecting a and b of length 2 or 3. We claim that if d is in B_{ab} , then either

- $\{a, d\}$ is a 2-cut of G with its good-side being the path $acbd$, or
- $\{b, d\}$ is a 2-cut of G with its good-side being the path $bcad$.

That is, if the good-sides of $\{a, b\}$ and $\{c, d\}$ intersect nontrivially, then both are contained in a larger good-side of another 2-cut.

Now we prove the claim. Suppose that d is in B_{ab} . Note that c is a cut-vertex of $G \setminus d$ and c has only two neighbors in $G \setminus d$ (same as in G), one is connected to a and the other is connected to b . Hence the graph $G \setminus \{c, d\}$ does not have a path between a and b , so that its subgraph $\tilde{B}_{ab} \setminus d$ does not have one either. Thus d is a cut-vertex of \tilde{B}_{ab} separating a and b . Moreover, $\tilde{B}_{ab} \setminus d$ has precisely two components, whose vertex sets we call D_a and D_b with $a \in D_a$ and $b \in D_b$; otherwise G is not 2-connected.

By Corollary 3.9, $ab \notin E(G)$ and $cd \notin E(G)$. Suppose that $D_a = \{a\}$. Since \tilde{B}_{ab} contains at least seven vertices, the set $\{d, b\}$ must be a 2-cut of G . The good-side of $\{d, b\}$, by counting the vertices, must be the component containing c and by Corollary 3.9 we deduce that $dacb$ is an induced path in G and a, c have no other neighbors, which is what we claimed. Similarly we can settle the case when $D_b = \{b\}$.

Thus we may assume that $|D_a| \geq 2$ and $|D_b| \geq 2$. Since $\{c, d\}$ is a 2-cut of G , by Corollary 3.9, we have either $|D_a| = 2$ or $|D_b| = 2$. We assume $|D_b| = 2$. As $D_a \cup D_b \cup \{d\}$ constitutes \tilde{B}_{ab} , we have $|D_a| + |D_b| + 1 \geq 7$, so that $|D_a| \geq 4$. Now we consider the 2-cut $\{a, d\}$. The component containing c and b has at least three vertices, so this component must be B_{ad} . But $|D_a| \geq 4$ implies $G \setminus \tilde{B}_{ad}$ having at least three vertices, a contradiction to Corollary 3.9. This proves the claim.

For a 2-cut $\{a, b\}$ of G , we denote by $C_{ab} = G \setminus B_{ab}$ the good-side of $\{a, b\}$. What we showed above is that, if $C_{ab} \cap C_{cd} \neq \emptyset$ for some $\{c, d\}$ then either (1) $C_{ab} \subseteq C_{cd}$, (2) $C_{cd} \subseteq C_{ab}$, or (3) the intersection consists of a single vertex which is one of a, b, c, d .

Now we replace the good-sides at once by edges between their cuts. Formally, let $\mathcal{C} = \{C_{ab} : \text{is a 2-cut of } \{a, b\} \text{ in } G\}$. Let $\mathcal{C}^* \subseteq \mathcal{C}$ be the subset of inclusion-wise maximal elements. For each $C_{ab} \in \mathcal{C}^*$, we remove $C_{ab} \setminus \{a, b\}$ from G and add an edge ab to G . This operation is well-defined because of the above discussion. Let H be the resulting proper minor of G . If H has a 2-cut, then we construct G back from H and the 2-cut still remains in G , which is impossible since for each 2-cut S , we remove all but one component of $G \setminus S$ while producing H . That is, H is either 3-connected or isomorphic to K_2 or K_3 .

If H has at least 7 vertices, then H has a minor in \mathcal{O}_3 so that $G \notin \mathcal{O}_2$. Thus H has at most 6 vertices. If H is K_2 , then H is constructed by connecting several good-sides in parallel, which has mm-width at most 2 as shown in the proof of Lemma 3.7, especially Fig. 3. If H is a triangle abc , then G is obtained from abc by good-subdividing all three edges ab, bc and ca . To find a tree-representation of G with width 2 in this case, we start from a $K_{1,3}$ where its three edges are labeled respectively by ab, bc and ca . Then we can add the good-sides for the edges ab, bc and ca without increasing the width. Hence H has 4, 5, or 6 vertices and is 3-connected. \square

The obstructions obtained from a 3-connected graph on 4, 5, and 6 vertices respectively are listed in Figs. 5, 6 and 7. The respective proofs are given in Lemmas 3.14, 3.15 and 3.16. Note that Lemmas 3.5 and 3.8 imply the following.

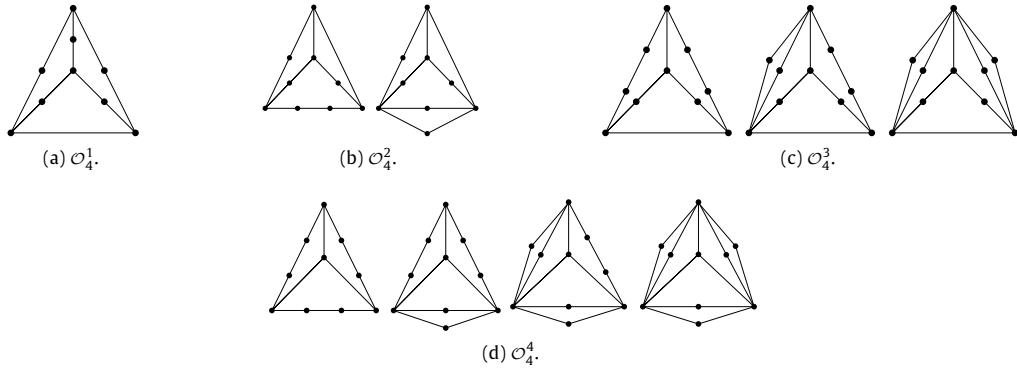


Fig. 5. The graphs in $\mathcal{O}_4 = \mathcal{O}_4^1 \cup \mathcal{O}_4^2 \cup \mathcal{O}_4^3 \cup \mathcal{O}_4^4$.

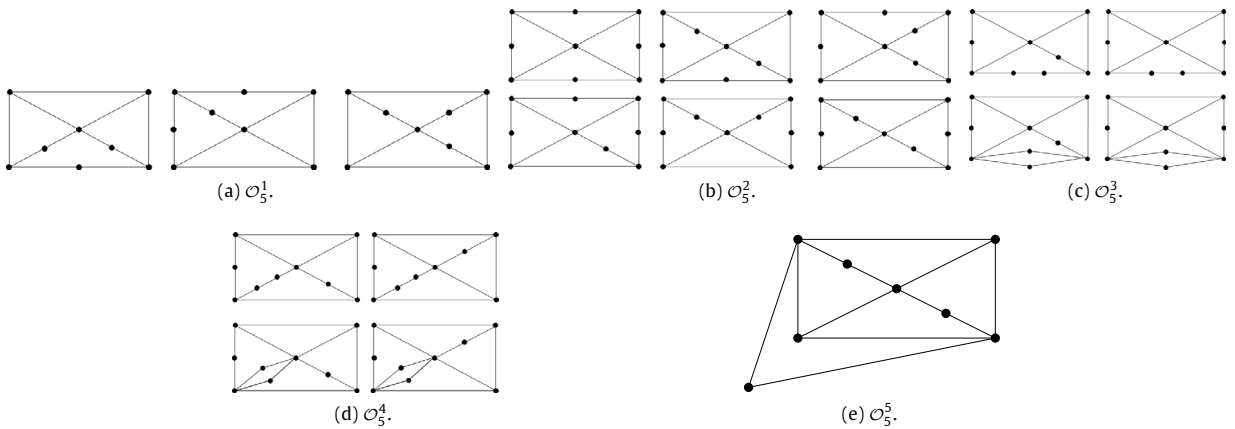


Fig. 6. The graphs in $\mathcal{O}_5 = \mathcal{O}_5^1 \cup \mathcal{O}_5^2 \cup \mathcal{O}_5^3 \cup \mathcal{O}_5^4 \cup \mathcal{O}_5^5$.

Lemma 3.11. Let G be a graph with an induced path $acdb$ such that c and d are non-adjacent to other vertices. Let H be the graph obtained from $G - \{c, d\}$ by adding two new vertices c', d' and paths $ac'b$ and $ad'b$. Then $G \in \mathcal{O}_2$ if and only if $H \in \mathcal{O}_2$.

Hence in the following discussion we do not consider 11-subdivisions. The obstructions obtained by replacing 2-subdivisions with 11-subdivisions shall be added to the list without mentioning.

We shall use the following lemma often when we show a graph has mm-width at most 2.

Lemma 3.12. Let G be a graph. If $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ and G has six vertices a_i, b_i for $i = 1, 2, 3$ such that for each i ,

- (1) $a_i, b_i \in V_i$,
- (2) $\{a_i, b_i\}$ separates V_i from $V(G) \setminus V_i$, and
- (3) $(G[V_i], \{a_i, b_i\})$ has a good tree-representation,

then $\text{mmw}(G) \leq 2$.

Proof. For each i , we consider a good tree-representation of $G[V_i]$ such that the subtrees for a_i and b_i share an edge whose one end has degree 1. We combine the three tree-representations by identifying those degree 1 vertices to obtain a tree-representation of G with width at most 2. \square

The way we use Lemma 3.10 to show a graph has mm-width ≤ 2 is that, we try to cover the graph with either three good-sides or two good-sides and a set of at most two vertices. If we do so, the sets become V_1, V_2 and V_3 in the statement and Lemma 3.10 applies.

For convenience, we state here at once that the graphs in the following Lemmas 3.14–3.16 all have mm-width at least 3. Lemma 2.6 is a corollary of the following lemma.

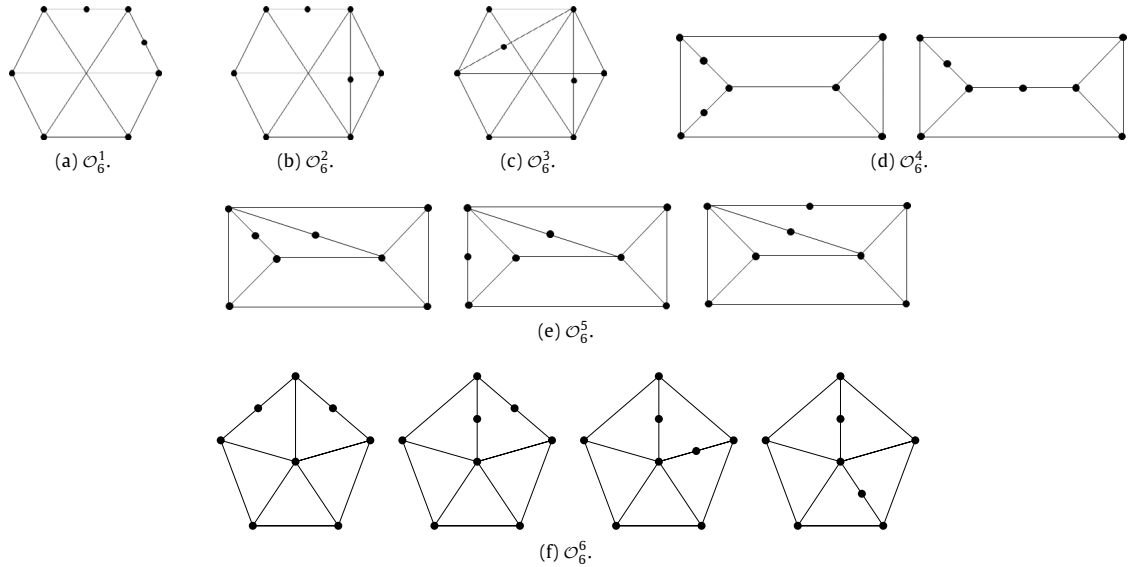


Fig. 7. The graphs in $\mathcal{O}_6 = \mathcal{O}_6^1 \cup \mathcal{O}_6^2 \cup \mathcal{O}_6^3 \cup \mathcal{O}_6^4 \cup \mathcal{O}_6^5 \cup \mathcal{O}_6^6$.

Lemma 3.13. Let \mathcal{O}_4 , \mathcal{O}_5 and \mathcal{O}_6 , respectively, be the set of graphs in Figs. 5–7. Every graph in $\mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$ has maximum matching width at least 3.

Proof. By Theorem 2.5, it is enough to give a tangle of order 3. We shall explain how to find a tangle of order 3 for each of those graphs.

Let G be a graph and let X be a subset of $V(G)$ such that $\text{mm}(X) \leq 2$ and $|X| \geq 3$. In other words, the bipartite graph on $V(G)$ with all the edges in $E(G)$ having one end in X and the other not in X has maximum matching size 2. Thus we can find a set $\{a, b\}$ that is a 2-cut of G and $G \setminus \{a, b\}$ has a component, say S , such that \bar{S} contains either X or $V(G) \setminus X$.

Therefore, for each graph $G \in \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$, we set S_G to be the collection of all vertex subsets of the following three types:

- a set of size at most 2
- a good-side of a 2-cut
- if G has a 11-subdivision made of the paths aub and avb , then S_G contains both $\{a, u, b\}$ and $\{a, v, b\}$.

Now we consider the tangle axioms (T1), (T2) and (T3) in Section 2.2 to verify that S_G is a tangle. (T1) follows immediately from the above discussion, and (T3) is also easy to check for all graphs in $G \in \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$. For (T2), we can check that no three good-sides cover the whole graph and it remains to see that there are no two good-sides that cover all but at most two vertices. We leave the details to the reader. \square

Now we consider the case when the 3-connected graph in Lemma 3.10 has four vertices. The only 3-connected graph on four vertices is K_4 .

Lemma 3.14. Let $\mathcal{O}_4^1, \mathcal{O}_4^2, \mathcal{O}_4^3, \mathcal{O}_4^4$ be the sets of graphs in Fig. 5. If a graph G is obtained from K_4 by good-subdividing some of its edges, then $G \in \mathcal{O}_2$ if and only if $G \in \mathcal{O}_4 = \mathcal{O}_4^1 \cup \mathcal{O}_4^2 \cup \mathcal{O}_4^3 \cup \mathcal{O}_4^4$.

Proof. By Lemma 3.13 the graphs in \mathcal{O}_4 have mm-width at least 3. It can be easily checked that all their proper minors have mm-width at most 2 using Lemma 3.12.

Now we consider the graphs obtained from K_4 by good-subdivisions. We divide the cases via the number of good-subdivisions. Recall that by Lemma 3.11, we only consider 2-subdivision and not 11-subdivision.

If G has no 2-subdivision and has at most four 1-subdivisions, then $\text{mmw}(G) \leq 2$ by Lemma 3.12. The unique graph with no 2-subdivision and five 1-subdivisions is in \mathcal{O}_4^1 .

If G has one 2-subdivision and at most three 1-subdivisions, then $\text{mmw}(G) \leq 2$ by Lemma 3.12 unless G is the first graph in \mathcal{O}_4^2 . If G has one 2-subdivision and four 1-subdivisions, then G contains the graph in \mathcal{O}_4^1 as a minor.

If G has two 2-subdivisions and at most two 1-subdivisions, then either G has mm-width 2, G contains a graph in \mathcal{O}_4^2 as a minor, or G is the first graph in \mathcal{O}_4^3 . The rest of \mathcal{O}_4^3 is obtained by replacing 2-subdivisions with 11-subdivisions; see Lemma 3.11. If G has more than two 1-subdivisions, then G contains the graph in \mathcal{O}_4^1 as a minor.

If G has three 2-subdivisions and no 1-subdivision, then $\text{mmw}(G) \leq 2$ by Lemma 3.12 unless G is the first graph in \mathcal{O}_4^4 . If G has three 2-subdivisions and at least one 1-subdivision, then G contains a graph in $\mathcal{O}_4^2 \cup \mathcal{O}_4^3$ as a minor. \square

Lemmas 3.15 and 3.16, respectively, characterize the graphs in \mathcal{O}_2 that are obtained from a 3-connected graph on five and six vertices.

Lemma 3.15. Let $\mathcal{O}_5^1, \mathcal{O}_5^2, \mathcal{O}_5^3, \mathcal{O}_5^4, \mathcal{O}_5^5$ be the sets of graphs in Fig. 6. If a graph G is obtained from a 3-connected graph on 5 vertices by good-subdividing some edges, then $G \in \mathcal{O}_2$ if and only if $G \in \mathcal{O}_5 = \mathcal{O}_5^1 \cup \mathcal{O}_5^2 \cup \mathcal{O}_5^3 \cup \mathcal{O}_5^4 \cup \mathcal{O}_5^5$.

Proof. By Lemma 3.13 the graphs in \mathcal{O}_5 have mm-width at least 3. Their proper minors have mm-width 2 by Lemma 3.12 and hence they are in \mathcal{O}_2 .

Now we consider the graphs that are also obtained from a 3-connected graph on 5 vertices by good-subdivisions. There are three 3-connected graphs on 5 vertices, namely the wheel W_5 , W_5 plus an edge (say W_5'), and K_5 .

Let us begin with W_5 . Let G be a graph obtained from W_5 by good-subdividing some edges.

Suppose that G has no 2-subdivision and has three 1-subdivisions. If the to-be-subdivided edges of W_5 contain two independent edges, then Lemma 3.12 implies $\text{mmw}(G) \leq 2$. Thus $G \in \mathcal{O}_2$ if and only if $G \in \mathcal{O}_5^1$. If G has no 2-subdivision and has four 1-subdivisions, then it has mm-width 3; a tangle of order 3 can be found in each case as in Lemma 3.13. So in this case $G \in \mathcal{O}_2$ if and only if it does not have a graph in \mathcal{O}_5^1 as a minor. These are the graphs in \mathcal{O}_5^2 .

If the number of 1-subdivisions and 2-subdivisions in G is at least 4, then G contains a graph in $\mathcal{O}_5^1 \cup \mathcal{O}_5^2$ as a minor. Suppose G has one 2-subdivision and two 1-subdivisions. If the good-side of the 2-subdivision does not intersect with one of the other two good-sides, then Lemma 3.12 implies $\text{mmw}(G) \leq 2$. Thus both good-sides of the 1-subdivisions intersect with the good-side of the 2-subdivision. If the 2-subdivision happens at an edge incident with the vertex of degree 4 in W_5 , then $G \in \mathcal{O}_2$ if and only if G is one of the top two graphs in \mathcal{O}_5^4 ; other cases contain a graph in \mathcal{O}_5^1 as a minor. The bottom two graphs in \mathcal{O}_5^4 are obtained by replacing the 2-subdivision with a 11-subdivision. If the 2-subdivision is not incident with the degree-4 vertex of W_5 , then we get the graphs in \mathcal{O}_5^3 .

If G has at least two 2-subdivisions, then either $\text{mmw}(G) \leq 2$ or it contains a graph in $\mathcal{O}_5^1 \cup \mathcal{O}_5^2 \cup \mathcal{O}_5^3 \cup \mathcal{O}_5^4$ as a minor. It completes the graphs obtained from W_5 .

Now we consider the graphs $G \in \mathcal{O}_2$ obtained from W_5' by good-subdivisions. The graph W_5' has three edges whose removal results in W_5 . Suppose one of these three edges, say e , is not good-subdivided in G . If G has at least four good-subdivisions, then $G - e$ contains a graph in $\mathcal{O}_5^1 \cup \mathcal{O}_5^2 \cup \mathcal{O}_5^3 \cup \mathcal{O}_5^4$ as a minor. If G has at most three good-subdivisions and $G - e$ does not contain a graph in \mathcal{O}_5^1 as a minor, then Lemma 3.12 implies $\text{mmw}(G) \leq 2$. Hence all three edges of W_5' in the triangle of degree-4 vertices must be good-subdivided in G . Since the graph in \mathcal{O}_5^5 is in \mathcal{O}_2 , it is the unique graph obtained from W_5' in \mathcal{O}_2 .

The last 3-connected graph on five vertices is K_5 . Let G be a graph obtained from K_5 by good-subdivisions. Using an argument similar to above we can show that every edge of K_5 must be subdivided. Hence $G \notin \mathcal{O}_2$ and \mathcal{O}_5 is the precise set of obstructions obtained from a 3-connected graph on five vertices. \square

Lemma 3.16. Let $\mathcal{O}_6^1, \mathcal{O}_6^2, \mathcal{O}_6^3, \mathcal{O}_6^4, \mathcal{O}_6^5, \mathcal{O}_6^6$ be the sets of graphs in Fig. 7. If a graph G is obtained from a 3-connected graph on 6 vertices by good-subdivisions, then $G \in \mathcal{O}_2$ if and only if $G \in \mathcal{O}_6 = \mathcal{O}_6^1 \cup \mathcal{O}_6^2 \cup \mathcal{O}_6^3 \cup \mathcal{O}_6^4 \cup \mathcal{O}_6^5 \cup \mathcal{O}_6^6$.

Proof. Let H be a 3-connected graph on six vertices and let G be a graph obtained from H by good-subdividing some edges. If two adjacent edges of H are good-subdivided in G , then we can find a tangle of order 3 in G and hence $\text{mmw}(G) \geq 3$; all graphs in \mathcal{O}_6 are of this type. If there is no such pair in H , then the good-subdivisions happened at a matching of H and Lemma 3.12 implies $\text{mmw}(G) \leq 2$. We leave it to the reader to check that the proper minors of the graphs in \mathcal{O}_6 have mm-width at most 2.

If H is minimally 3-connected, then all the graphs obtainable from H by good-subdividing two adjacent edges are in \mathcal{O}_6 ; \mathcal{O}_6^1 for $K_{3,3}$, \mathcal{O}_6^4 for the prism and \mathcal{O}_6^6 for the wheel W_6 .

Suppose that H is not minimally 3-connected and $G \in \mathcal{O}_2$. Let e be an edge of H such that $H - e$ is still 3-connected. If e is not subdivided in G , then by the above discussion $G - e$ has two adjacent good-sides and $\text{mmw}(G - e) \geq 3$, a contradiction. Thus e must be good-subdivided in G and H has at most two edges whose removal does not affect its 3-connectivity. Note that if H has two such edges, then they should be also adjacent.

If H is $K_{3,3}$ plus an edge, then the additional edge must be subdivided and we need another adjacent edge to subdivide. But independently of this choice the resulting graph is isomorphic to the graph in \mathcal{O}_6^2 . There is a unique way of adding two adjacent edges to $K_{3,3}$ and the graph in \mathcal{O}_6^3 is the result of subdividing both.

If H is the prism plus an edge, then we have three non-isomorphic choices of another adjacent edge to subdivide. They are in \mathcal{O}_6^5 . There is again a unique way of adding two adjacent edges to the prism but it contains a graph in \mathcal{O}_6^6 as a minor.

There is a unique (up to isomorphism) way to add an edge to W_6 but it already has three edges that are removable while maintaining 3-connectivity. Thus the list is complete. \square

By Lemma 3.4, a graph is in the obstruction set and 3-connected if and only if it is in \mathcal{O}_3 . If G is in the obstruction set but not 3-connected, then it should be obtained from a 3-connected graph on 4, 5, or 6 vertices by Lemma 3.10. Lemmas 3.14–3.16 show that $G \in \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$. Therefore, the following theorem holds:

Theorem 3.17. Let $\mathcal{O} = \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$ be the set of 45 graphs in Figs. 1, 5–7. A graph G has maximum matching width at most 2 if and only if G has no minor isomorphic to a graph in \mathcal{O} .

4. $k \times k$ -grid

The $k \times k$ -grid, denoted by G_k , is the graph with a vertex set $V(G_k) = \{(i, j) : 1 \leq i, j \leq k\}$ and an edge set $E(G_k) = \{(i, j)(i', j') : |i - i'| + |j - j'| = 1\}$. In this section, we show $\text{mmw}(G_k) = k$ for $k \geq 2$.

Vatschelle [20] showed the following inequality. Recall that $\text{rw}(G)$ and $\text{brw}(G)$ respectively denote the rank-width and the branch-width of G .

Theorem 4.1 ([20]). If G is a graph, then

$$\text{rw}(G) \leq \text{mmw}(G) \leq \max(\text{brw}(G), 1).$$

It is known that $\text{brw}(G_k) = k$ [12] and $\text{rw}(G_k) = k - 1$ [7]. Hence $\text{mmw}(G_k)$ is either $k - 1$ or k . We shall show $\text{mmw}(G_k) > k - 1$ by finding a *tangle* of order k ; see Section 2.2. We assume $k \geq 2$ throughout this section.

Let $C_i = \{(i, j) : 1 \leq j \leq k\}$ and $R_j = \{(i, j) : 1 \leq i \leq k\}$ be the set of vertices on the i -th column and the j -th row respectively. Recall that for a vertex set $X \subseteq V(G)$, $\text{mm}_G(X)$ denotes the size of a maximum matching in $G[X, V(G) \setminus X]$. We omit G_k in mm_{G_k} and write $\text{mm}(X) = \text{mm}_{G_k}(X)$ in this section. Let $X^c = V(G_k) \setminus X$ for $X \subseteq V(G_k)$.

Lemma 4.2. If $X \subseteq V(G_k)$ and $\text{mm}(X) < k$, then $R_i \subseteq X$ for some i if and only if $C_j \subseteq X$ for some j .

Proof. Suppose that $R_i \subseteq X$ for some i . Then each C_j intersects with X . If $C_j \not\subseteq X$ for every j , each $G[C_j]$ contains an edge with one end in X and the other end in X^c . Since these edges form a matching of size k , we have $\text{mm}(X) \geq k$ which is a contradiction. Thus $C_j \subseteq X$ for some j . The converse follows from the symmetry. \square

For $X \subseteq V(G_k)$, we say that X is *small* if $\text{mm}(X) < k$ and $R_i \not\subseteq X$ for all $i = 1, 2, \dots, k$. Note that, by Lemma 4.2, $C_j \not\subseteq X$ for all $j = 1, 2, \dots, k$ if X is small.

Lemma 4.3. Let $X \subseteq V(G_k)$. If $\text{mm}(X) < k$, then one of X and X^c is small.

Proof. Suppose that neither X nor X^c is small. Then we can choose i_1, i_2 with $1 \leq i_1, i_2 \leq k$ such that $R_{i_1} \subseteq X$ and $R_{i_2} \subseteq X^c$. Now we may choose an edge from each column of G_k with endpoints one in X and the other in X^c . Since these edges form a matching of size k , we have $\text{mm}(X) \geq k$, a contradiction. \square

Lemma 4.4. If $X \subseteq V(G_k)$ is small, then there exist i, j such that $R_i \cap X = C_j \cap X = \emptyset$.

Proof. Suppose that $|R_i \cap X| > 0$ for all i . Since X is small, $R_i \cap X^c \neq \emptyset$. Thus, $G[R_i]$ contains an edge between X and X^c for every i . These edges show that $\text{mm}(X) \geq k$, a contradiction. Likewise, $C_j \cap X = \emptyset$ for some j . \square

Lemma 4.5. If $X_1 \cup X_2 \cup X_3 = V(G_k)$, then one of X_1, X_2 , and X_3 is not small.

Proof. We prove by induction on k . The lemma is trivial when $k = 2$. Assume that $k > 2$ and the lemma is true for $k - 1$. To prove by contradiction, let us suppose that all of X_1, X_2 , and X_3 are small. Note that each row or column intersects at least two of X_1, X_2 and X_3 .

Firstly we suppose that $R_k \cup C_k$ intersects X_t for all $t \in \{1, 2, 3\}$. We consider the $(k - 1) \times (k - 1)$ -grid $G_{k-1} = G_k \setminus (R_k \cup C_k)$ with sets $X'_t = X_t \setminus (R_k \cup C_k)$ for each $t \in \{1, 2, 3\}$ so that $X'_1 \cup X'_2 \cup X'_3 = V(G_{k-1})$. By the induction hypothesis, we may assume that X'_1 is not small in G_{k-1} . That is, $\text{mm}_{G_{k-1}}(X'_1) \geq k - 1$ or X'_1 contains a row of G_{k-1} . If $\text{mm}_{G_{k-1}}(X'_1) \geq k - 1$, then G_{k-1} has a matching of size $k - 1$ between X'_1 and $V(G_{k-1}) \setminus X'_1$. Since G_k has an edge in $G_k[R_k \cup C_k]$ with one end in X_1 and the other in X_1^c , we obtain a matching of size k in $G_k[X_1, X_1^c]$ showing that $\text{mm}(X_1) \geq k$ and X_1 is not small. Hence we may assume that $\text{mm}_{G_{k-1}}(X'_1) < k - 1$ and X'_1 contains a row R' of G_{k-1} . Since we assumed X_1 to be small, one of the columns of G_k does not intersect X_1 by Lemma 4.4 but it must be C_k ; all other columns intersect with R' . On the other hand, by Lemma 4.2, X'_1 also contains a column of G_{k-1} and R_k does not intersect X_1 . Thus $(R_k \cup C_k) \cap X_1 = \emptyset$, a contradiction to our assumption that $R_k \cup C_k$ intersects all of X_1, X_2 and X_3 .

Therefore we may assume that for every choice $i, j \in \{1, k\}$, $R_i \cup C_j$ does not intersect all X_t at the same time. Since each row and column intersects at least two of X_1, X_2 and X_3 , if $R_1 \cup R_k$ meets all X_t , then either $R_1 \cup C_k$ or $R_k \cup C_k$ meets all X_t so that we assume each of R_1 and R_k intersects X_1 and X_2 but not X_3 . It follows also that each of C_1 and C_k intersects X_1 and X_2 but not X_3 .

We shall show $\text{mm}(X_1) + \text{mm}(X_2) \geq 2k$ by proving that each column of G_k contains either two independent edges from one of $E[X_1, X_1^c]$ and $E[X_2, X_2^c]$, or one edge from each set. Those edges form two matchings in $G[X_1, X_1^c]$ and $G[X_2, X_2^c]$ respectively whose sizes sum up to at least $2k$. Thus we get $\text{mm}(X_1) \geq k$ or $\text{mm}(X_2) \geq k$ and one of X_1 and X_2 is not small.

If a column has an edge with one end in $X_1 \setminus X_2$ and the other in $X_2 \setminus X_1$ then we are done. Thus C_1 and C_k are fine. If all columns are as such then we are done. Otherwise, there is a column C_i such that $C_i \cap X_2 \subseteq C_i \cap X_1$. Since $C_i \not\subseteq X_1$, we have $|C_i \cap (X_3 \setminus X_1)| > 0$. If $|C_i \cap (X_3 \setminus X_1)| \geq 2$ then C_i has two independent edges in $E[X_1, X_1^c]$. Thus we assume $|C_i \cap (X_3 \setminus X_1)| = 1$, that is, $|C_i \cap X_1| = k - 1$. By Lemma 4.4 we choose a column C_j not intersecting with X_1 , and between C_i and C_j we can find $k - 1$ independent row-edges in $E[X_1, X_1^c]$. Since C_1 and C_k are not in this area, we may choose an edge from $G[C_1] \cap G[X_1, X_1^c]$ and $G[X_1, X_1^c]$ has a matching of size k , showing that $\text{mm}(X_1) \geq k$ and X_1 is not small. This final contradiction completes the proof. \square

Lemma 4.6. *Let \mathcal{T} be the set of all small subsets of $V(G_k)$. The set \mathcal{T} is a tangle in G_k of order k .*

Proof. The first and second axioms follow from Lemmas 4.3 and 4.5 respectively. For each $x \in V(G_k)$, the set $V(G_k) \setminus \{x\}$ contains a row and thus not in \mathcal{T} . \square

By Theorem 2.5, Lemma 4.6 implies $\text{mmw}(G_k) > k - 1$. Since the branchwidth of G_k is k , by Theorem 4.1, $\text{mmw}(G_k)$ is at most k .

Theorem 4.7. *The $k \times k$ -grid has maximum matching width k for $k \geq 2$.*

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